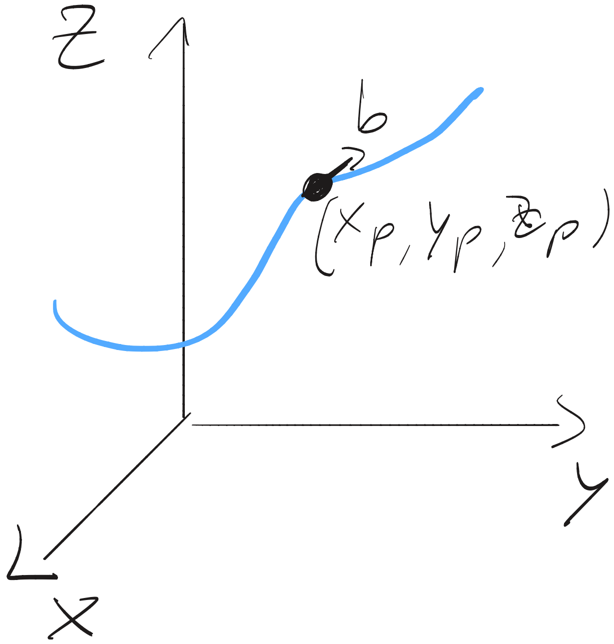
Review of some Fluid kinematic: Reynolds Transport Theorem & Fundamental Equations

Recalling some key concepts

1) MATERIAL DERIVATIVE



$$\vec{b} = b(x_p, y_p, z_p, t)$$

generic characteristic of that specific particle in a specific point

$P: x_p, y_p, z_p$

that is moving in a flow field $\vec{V} = V(u, v, w)$

HOW DOES THAT CHANGE in time?

$$\frac{d\vec{b}}{dt} = \frac{d}{dt} b(x_p(t), y(t), z(t), t)$$

$$= \frac{\partial b}{\partial t} + \frac{\partial b}{\partial x}\underbrace{\frac{dx_p}{dt}}_{u} + \frac{\partial b}{\partial y}\underbrace{\frac{dy_p}{dt}}_{v} + \frac{\partial b}{\partial z}\underbrace{\frac{dz_p}{dt}}_{w}$$

$$= \frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} + w\frac{\partial b}{\partial z}$$

$$\frac{\partial b}{\partial t} + V \cdot \nabla b = \frac{Db}{Dt}$$ MATERIAL DERIVATIVE

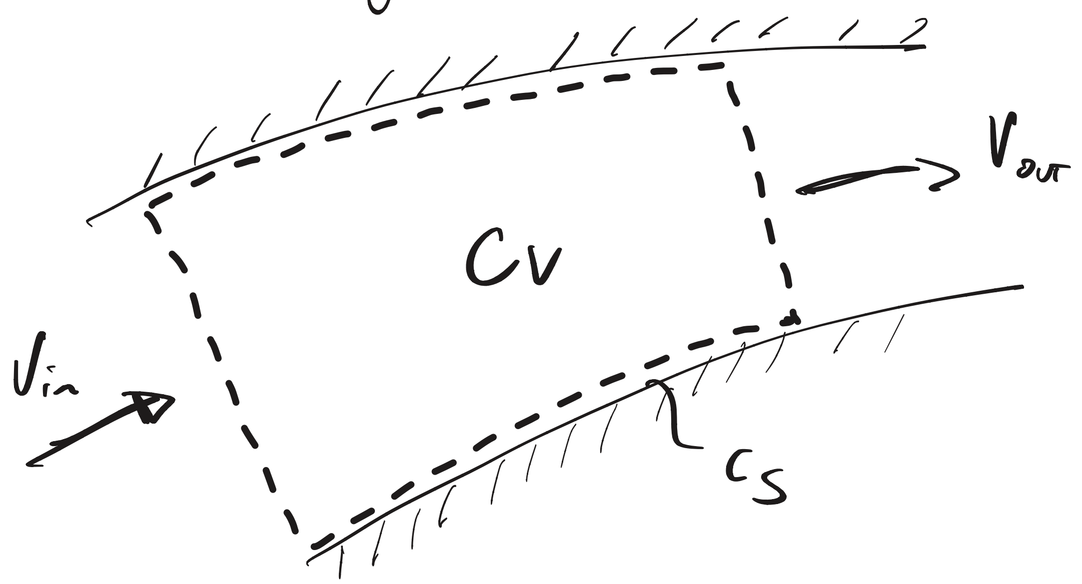
$\frac{\partial b}{\partial t}$: change of b with respect to (WRT) time

$V \cdot \nabla b$: change of b due to the moving flow.

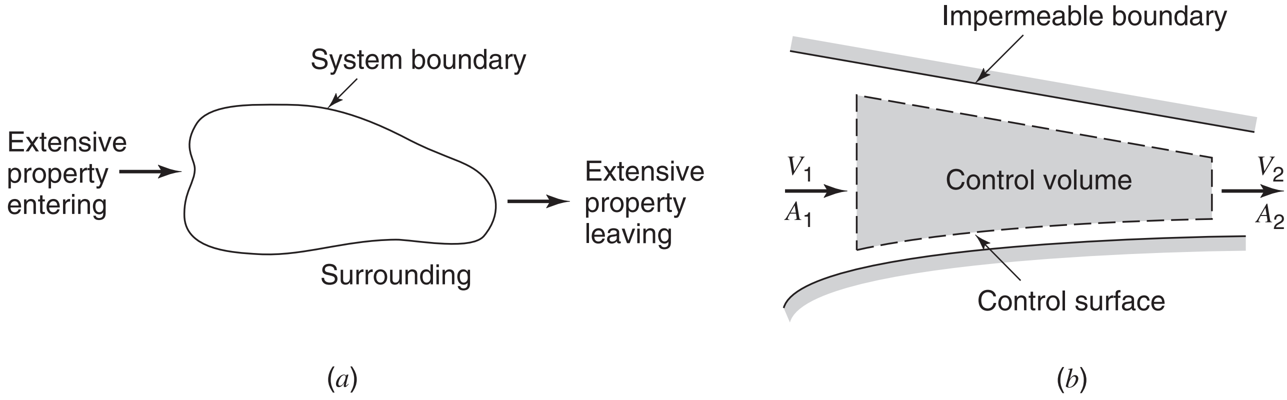
2) CONTROL Volume

is a volume in space (a geometric entity, independent of mass) through which fluid may flow

- control volume (CV) is a fix region in space
- control surface (CS) is the surface surrounding CV



Some useful definitions



System — a given quantity of matter. A system is also a set of connected parts that form a whole. In fluids: collection of fluid particles.

Control Volume (CV) — a volume in space

Control Surface (CS) — the surface of a control volume

In several demonstrations of fundamental equation in fluid mechanics, you take the CV to be coincidental with the system in a specific instant.

Extensive Property, B — physical property of a system [e.g., mass (m), momentum (mV), Energy (E), temperature, acceleration, etc.]

Intensive Property, b — the amount of property B per unit mass of matter (i.e., b=B/m). For example, the intensive properties that correspond to some of the extensive properties listed above are m/m=1, mV/m=V, E/m=e

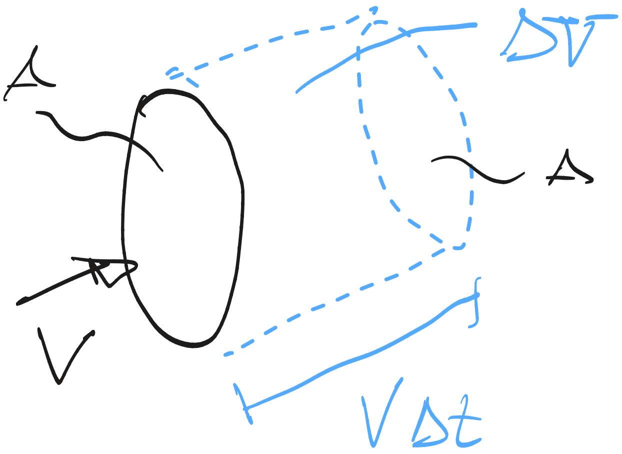
Uniform — having properties that are invariant in space

Steady — invariant with time, d(-)/dt=0. A steady flow is a flow that does not change in time.

V = velocity

$\bar{V}$ = volume

• DEFINITION OF VOLUME FLOW RATE



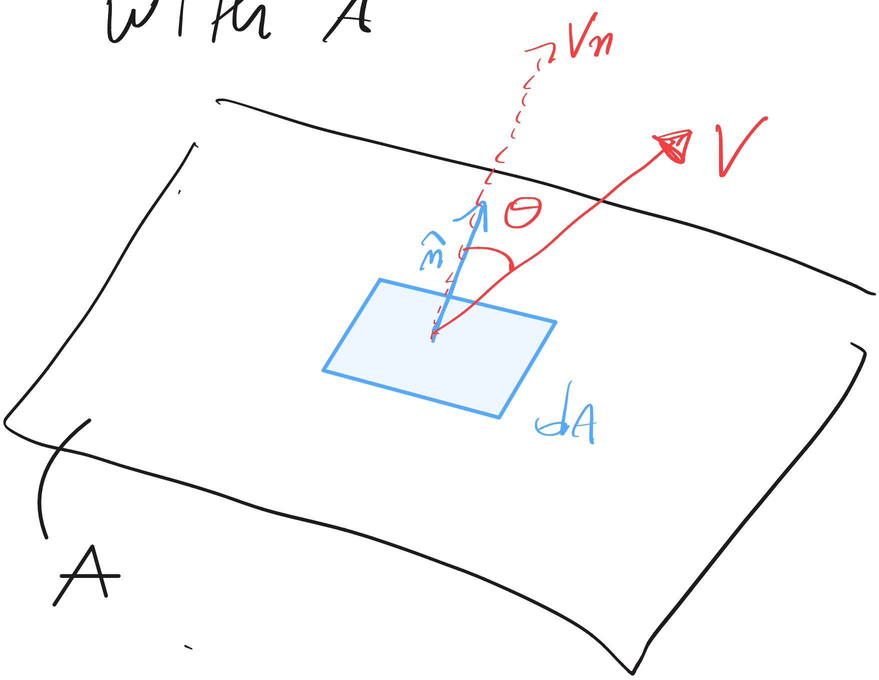
$$\text{Volume Change in time} = \frac{\Delta \bar{V}}{\Delta t} \quad \frac{[m^3]}{[s]}$$

$$= \frac{V \Delta t \cdot A}{\Delta t} = VA$$

if V = const across A & $\perp$ to A :

$$\Rightarrow \boxed{\text{FLOW DISCHARGE} \quad Q = VA}$$

IF V and A are not $\perp$ or V varies with A



$$V_n = |V| \cos\theta = \vec{V} \cdot \vec{n}$$

$$dQ = V_n \, dA$$

$$Q = \int_A dQ = \int_A V_n \, dA = \int_A \vec{V} \cdot \vec{n} \, dA$$

$$[Q] = [V][A] = \frac{m}{s} m^2 \Rightarrow \left[\frac{m^3}{s}\right]$$

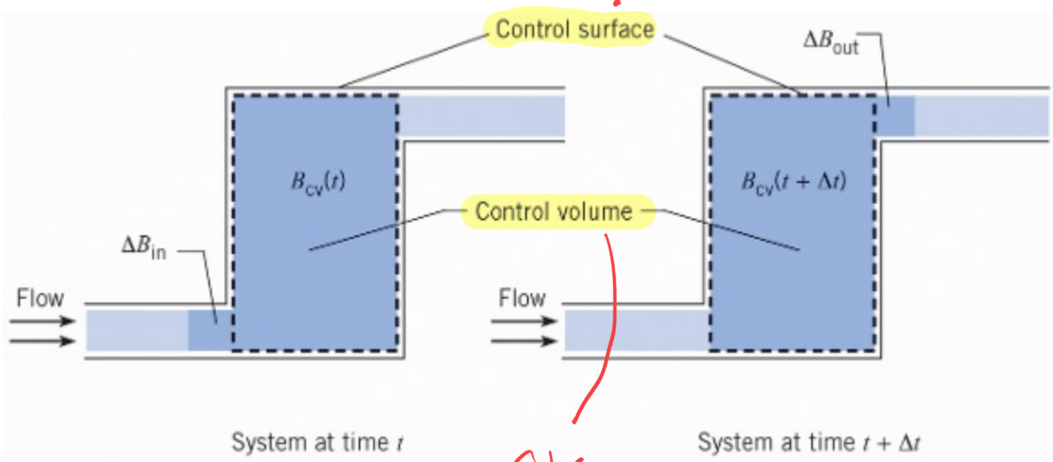
the mass flow rate

$$\dot{m} = \frac{\Delta m}{\Delta t} = \frac{\rho \Delta V}{\Delta t} = \rho Q = \rho A V$$

$$\left[\frac{kg}{m^3}\right]\left[m^2\right]\left[\frac{m}{s}\right] = \underline{\underline{\left[\frac{kg}{s}\right]}}$$

3) REYNOLDS TRANSPORT THEOREM

CS



CV

(This is shown in class through slides and it was already demonstrated in Fluid Mechanics)

Sometimes we may be interested in what effect a moving fluid has on a particular object or volume in space as the fluid interacts with it. To do so, we need to describe the laws governing fluid motion using both SYSTEM concepts (i.e., the mass of fluid seen in a Lagrangian way) and the CONTROL VOLUME concept (i.e., given volume in space, like expressed before, and thus in an Eulerian way). The Reynolds Transport Theorem is the tool to shift from one representation to the other.

$\hookrightarrow$ (RTT)

Let's take a generic EXTENSIVE property of our fluid in motion

EXTENSIVE $\leftarrow$ $B = b \cdot m$ $\rightarrow$ INTENSIVE

- B is a physical parameter of the flow:

$$B = \begin{cases} m & \text{MASS} \\ mV & \text{MOMENTUM OF the MASS} \\ E & \text{ENERGY} \end{cases}$$

EXTENSIVE because it depends on the amount of fluid

- b is the INTENSIVE property $\Rightarrow$ per unit mass

$$b = \frac{B}{m} = \begin{cases} 1 \\ V \\ E/m = e \end{cases}$$

The RTT allow us to monitor B of the system in time

$\Downarrow$

$$\boxed{\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho b \, d\bar{V} + \int_{CS} \rho b \, \vec{V} \cdot \vec{n} \, dA}$$

VARIATION OF B as fluid is moving (LAGRANGIAN)

RATE OF CHANGE of B as fluid pass through (EULERIAN)

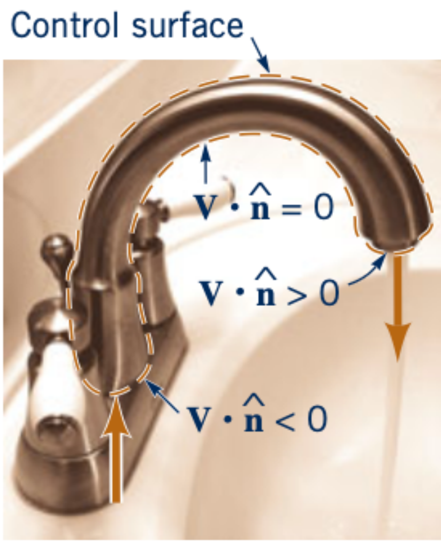
NET flux (or flowrate) of B across the entire control surface

• PHYSICAL INTERPRETATION

- The left side of the equation is the time rate of change of an arbitrary extensive parameter of a system. This may represent the rate of change of mass, momentum, energy, or angular momentum of the system, depending on the choice of the parameter B.

- recall that b is B per unit mass, so rho*b*dV is the amount of B in a small volume dV. Therefore, the first term on the right hand side is the rate of change of B within the control volume at any given time.

- The second term on the right hand side is the flow rate across the surface. The dot product between V and n indicates the direction of the flow: if V and n are opposite, the product is <0 and so the flow is entering the surface (the property B is carried inside); if V and n are in the same direction, the product is >0 and so the flow is exiting the control volume (the property is carried outside). See the example of the faucet here.



⇒ can you see the Similarity with the MATERIAL DERIVATIVE?

Application of the RTT: derivation of the fundamental equtions

1. Conservation of Mass — the Continuity Equation

A SYSTEM is defined es e COLLECTION OF UNCHANGING CONTENTS ⇒

$$\boxed{\text{CONSERVATION OF MASS PRINCIPLE}}$$

CONSERVATION OF MASS = Time rate of change of the system mass is = 0

⇓

$$\frac{Dm_{sys}}{Dt} = 0$$

where $m_{sys} = \int_{V} \rho \, dV$

i.e. sum of all the little elements of mass $dm = \rho \, dV$

For a system and a FIXED, non-deforming, CV that are COINCIDENT at an instant in time, we can use the RTT with MASS as the simplest extensive property

$$\Rightarrow \quad B = m \quad \Rightarrow \quad b = \frac{B}{m} = \frac{m}{m} = 1$$

Applying the REYNOLDS TRANSPORT THEOREM:

$$\underbrace{\left(\frac{Dm}{Dt}\right)_{sys} = \frac{D}{Dt}\int_{sys} \rho \, dV}_{\text{TIME RATE OF CHANGE OF THE MASS OF THE COINCIDENT SYSTEM}} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, dV}_{\text{TIME RATE OF CHANGE OF THE CONTENTS OF COINCIDENT CV}} + \underbrace{\int_{CS} \rho \vec{V} \cdot \hat{n} \, dA}_{\text{NET RATE OF FLOW THROUGH THE CONTROL SURFACE}}$$

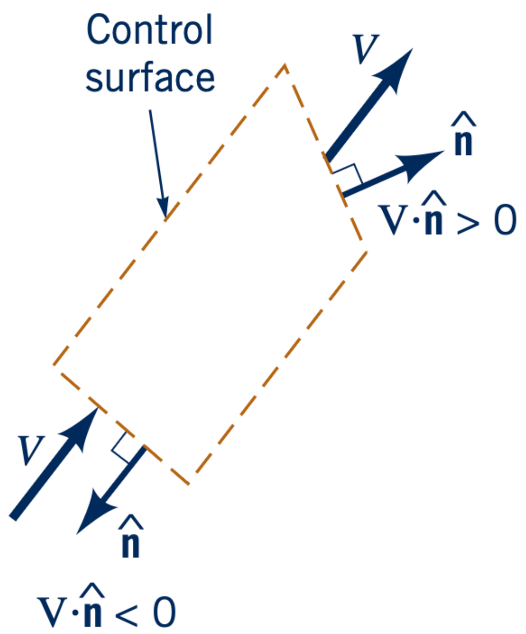
TIME RATE OF CHANGE OF THE MASS OF THE COINCIDENT SYSTEM = TIME RATE OF CHANGE OF THE CONTENTS OF COINCIDENT CV + NET RATE OF FLOW THROUGH THE CONTROL SURFACE

$\Rightarrow$ So the CONSERVATION OF mass lead to:

$$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot \hat{n} \, dA = 0}$$

CONTINUITY EQUATION!

Let's visualize the net flux component in a simpler way:



$\vec{V}\cdot\vec{n}\,dA$ is the volume flow rate through dA, while $\rho\vec{V}\cdot\vec{n}\,dA$ is the mass flow rate through dA

- $\vec{V}\cdot\vec{n}$ is (+) when flowing OUT
- $\vec{V}\cdot\vec{n}$ is (−) when flowing IN

So when all the differential quantities are summed over the entire surface (i.e., the INTEGRAL), the result is the NET MASS FLOW RATE IN & OUT of CS or also:

$$\boxed{\int_S \rho\vec{V}\cdot\vec{n}\,dA = \sum \dot{m}_{OUT} - \sum \dot{m}_{IN}}$$

So the continuity equation :

$$\boxed{\frac{\partial}{\partial t}\int_V \rho \, dV + \int_S \rho \vec{V}\cdot\vec{n} \, dA = 0}$$

OR

$$\boxed{\frac{\partial}{\partial t}\int_V \rho \, dV + \sum \dot{m}_{OUT} - \sum \dot{m}_{IN} = 0}$$

IF the problem is STEADY $\longrightarrow \frac{\partial}{\partial t} = 0$

$$\boxed{\Rightarrow \quad \sum \dot{m}_{OUT} - \sum \dot{m}_{IN} = 0}$$

when flow is uniformely distributed over the control surface

$$\dot{m} = \rho V A$$

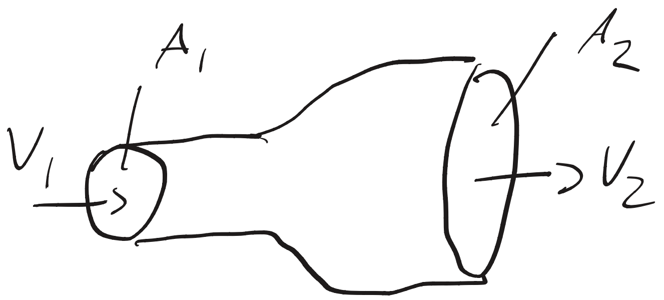
$$\Rightarrow \quad \boxed{\sum \rho_{OUT} V_{OUT} A_{OUT} - \sum \rho_{in} V_{in} A_{in} = 0}$$

for INCOMPRESSIBLE flows $\rho = \text{const}$

$$\Rightarrow \quad \sum V_{OUT} A_{OUT} - \sum V_{in} A_{in} = 0$$

$$\boxed{\sum Q_{out} - \sum Q_{in} = 0}$$

so in a pipe



$$\Downarrow$$

$$Q_1 = Q_2$$

$$\underline{V_1 A_1 = V_2 A_2}$$

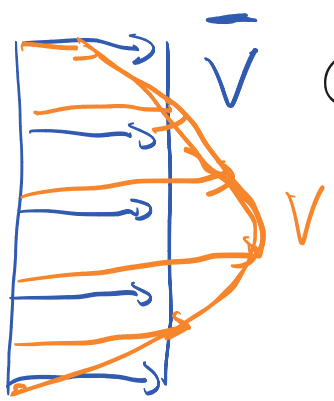
$\boxed{\text{NOTE!}}$

In general the mass flow rate is

$$\dot{m} = \int_A \rho \vec{V} \cdot \hat{n} \, dA$$

to get to $\dot{m} = \rho A V = \rho Q$ as seen before, ρ & V must be representative or the <u>AVERAGE VALUES</u>

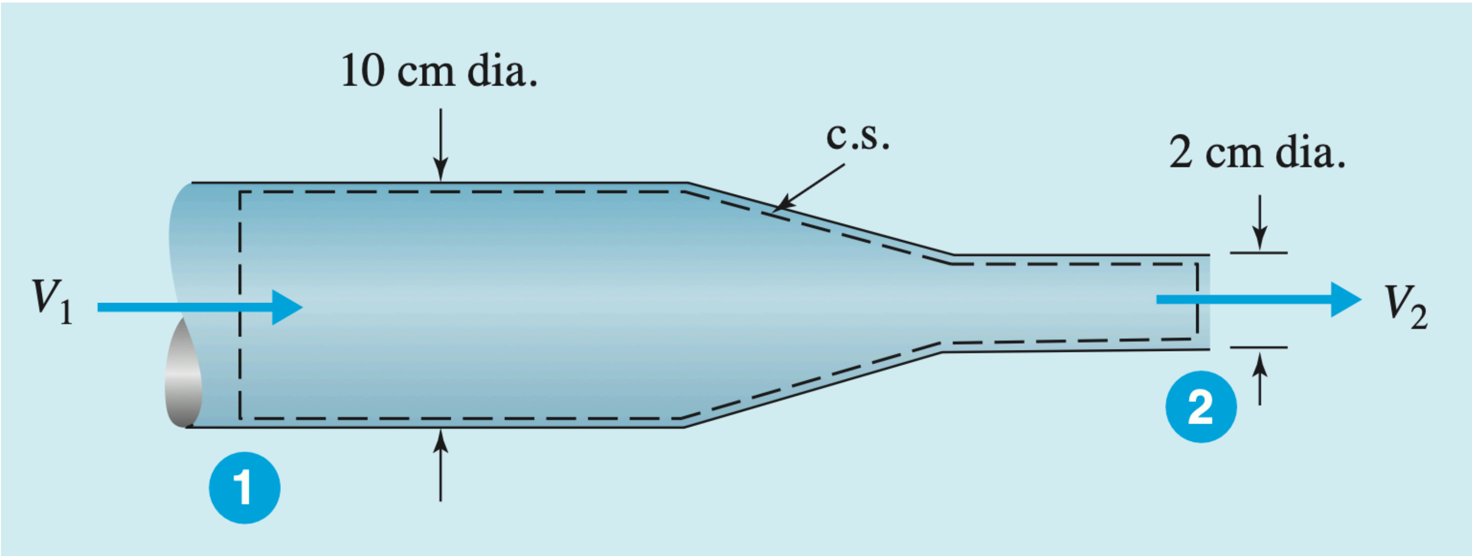
1. for incompressible flow $\rho = \text{const}$ ok!
2. V in $\dot{m}$ is the average $\bar{V}$, defined as



$$\bar{V} = \frac{\int_A \rho \vec{V} \cdot \hat{n} \, dA}{\rho A}$$

Example

Water flows at a uniform velocity of 3 m/s into a nozzle that reduces the diameter from 10 cm to 2 cm. Calculate the water's velocity leaving the nozzle and the flow rate



The control volume is selected to e the inside of the nozzle. Flow enters the CV at section 1 and leaves at section 2. We use the simplified continuity equation we just saw for uniform velocity profiles (assumption1) and constant water density (assumption 2):

$$A_1 V_1 = A_2 V_2$$

$$V_2 = V_1 \frac{A_1}{A_2} = 3 \frac{\pi \cdot 0.1^2/4}{\pi \cdot 0.02^2/4} = 75 \ m/s$$

The flow rate, or discharge Q, is then:

$$Q = V_1 A_1$$

$$= 3 \cdot \pi \cdot \frac{0.1^2}{4} = 0.0236 \ \frac{m^3}{s}$$

$$= V_2 A_2 \ \checkmark$$

2. First Law of Thermodynamics — the Energy Equation

The ENERGY EQUATION is derived applying the 1^{st} LAW OF THERMODYNAMICS along with the CV approach (i.e., RTT)

It's an energy balance that accounts for all INPUTS & OUTPUTS of ENERGY to and from the SYSTEM.

FIRST LAW OF THERMODYNAMICS = the rate of change of total stored energy of the SYSTEM in time is equal to the rate at which HEAT is transferred into the fluid $\left(\frac{dQ_H}{dt}\right)$ minus the rate at which the fluid does WORK $\left(\frac{dW}{dt}\right)$ on the surroundings.

$$\frac{dE}{dt} = \underbrace{\frac{dQ_H}{dt}}_{\dot{Q}} - \underbrace{\frac{dW}{dt}}_{\dot{W}}$$

$E \rightarrow$ TOTAL ENERGY of the system.

$$\Downarrow$$

$$E = E_u + E_k + E_p$$

where:

E_u = internal energy

E_k = KINETIC energy $= \frac{mV^2}{2}$

E_p = POTENTIAL energy $= mgz$

E is the extensive property of the system $= B_{sys}$

so the intensive property $b = \frac{B}{m}$:

$$b = \frac{B}{m} = e = e_u + e_k + e_p$$

where

$$e_k = \frac{\cancel{m}V^2/2}{\cancel{m}} = \frac{V^2}{2}$$

$$e_p = \frac{\cancel{m}gz}{\cancel{m}} = gz$$

So using the RTT:

$$\boxed{\frac{dE}{dt} = \dot{Q}_H - \dot{W} = \frac{\partial}{\partial t}\int_{CV} e\rho \, dV + \int_{CS} e\rho \vec{V}\cdot\hat{n}\, dA}$$

$$\Rightarrow \dot{Q}_H - \dot{W} = \frac{\partial}{\partial t}\int_{CV}\left(e_u + \frac{1}{2}V^2 + gz\right)\rho \, dV + \int_{CS}\left(e_u + \frac{1}{2}V^2 + gz\right)\rho \vec{V}\cdot\hat{n}\, dA$$

NOTE: in many engineering applications, the process is ADIABATIC $\Rightarrow \dot{Q} = 0$

WORK of a system divided in 2:

FLOW WORK (W_f)

the work associated with the fluid normal stresses over a DISTANCE

SHAFT WORK (W_s)

work associated to MECHANICAL MACHINES, like turbines, propellers, fans etc.

$\Downarrow$

Flow work is the result of pressure forces as the system moves through space and Shaft work is any other work beside the flow work

The fluid normal stress is the PRESSURE (P) from all the direction

The work is a $\boxed{\text{force}}$ times a DISTANCE "d"

* it's a DOT PRODUCT!

$$W_f = F \cdot d \Rightarrow \frac{dW_f}{dt} = F \cdot V$$

where the FORCE is $F = PA$

$$\Rightarrow \frac{dW_f}{dt} = \dot{W}_f = \int_{cs} P \vec{V} \cdot \hat{n} \, dA$$

$$\dot{W} = \dot{W}_s + \int_{cs} p\vec{V}\cdot\hat{n}\,dA$$

$$\dot{Q}_H - \dot{W}_s - \int_{cs} p\vec{V}\cdot\hat{n}\,dA = \frac{\partial}{\partial t}\int_{cv} \rho e\,d\forall + \int_{cs} \rho e\vec{V}\cdot\hat{n}\,dA$$

$$\underbrace{\dot{Q}_H - \dot{W}_s - \int_{cs} \frac{p}{\rho}\rho\vec{V}\cdot\hat{n}\,dA = \frac{\partial}{\partial t}\int_{cv}\left(e_u + \frac{V^2}{2} + gz\right)\rho\,dA + \int_{cs}\left(e_u + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot\hat{n}\,dA}$$

$$\boxed{\dot{Q}_H - \dot{W}_s = \underbrace{\frac{\partial}{\partial t}\int_{cv}\left(e_u + \frac{V^2}{2} + gz\right)\rho\,dA}_{\text{for STEADY FLOW this term} \to 0} + \int_{cs}\left(\frac{p}{\rho} + e_u + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot\hat{n}\,dA}$$

Let's look at the NET FLUX term:

$$\int_{cs}\underbrace{\left(e_u + \frac{V^2}{2} + \frac{p}{\rho} + gz\right)}\rho\vec{V}\cdot\hat{n}\,dA$$

IF ALL of these quantities are UNIFORMLY DISTRIBUTED over the cross section, they can be carried out of the integral and $\int \rho \vec{V} \cdot \hat{n}\, dA = \dot{m}$

So:

$$\int_{CS} \left(e_v + \frac{V^2}{2} + \frac{P}{\rho} + gz\right) \vec{V} \cdot \hat{n}\, dA = \sum_{\substack{\text{flow}\\ \text{OUT}}} \left(e_v + \frac{V^2}{2} + \frac{P}{\rho} + gz\right) \dot{m} -$$

$$- \sum_{\substack{\text{flow}\\ \text{in}}} \left(e_v + \frac{V^2}{2} + \frac{P}{\rho} + gz\right) \dot{m}$$

$\boxed{\text{NOTE!}}$

The velocity is rarely UNIFORMLY DISTRIBUTED ~~over~~ the cross-section.
So the term: $\int_{CS} \frac{V^2}{2} \rho \vec{V} \cdot \hat{n}\, dA$ requires special attention.
Namely, we relate it to the average velocity $\bar{V}$ using a multiplicative coefficient called KINETIC energy COEFFICIENT α

$$\frac{\dot{m}\, \alpha\, \vec{V}^2}{2} = \int_A \frac{V^2}{2} \rho \vec{V} \cdot \vec{n}\, dA$$

$$\boxed{\alpha = \frac{\int_A \frac{V^2}{2} \rho \vec{V} \cdot \hat{n}\, dA}{\dot{m}\, \vec{V}^2 / 2}}$$

using the AVERAGE TERM for $\dot m = \rho A \bar V$

$$\alpha = \frac{\int_A \frac{V^2}{2} \rho \vec V \cdot \hat n \, dA}{\rho A \frac{\bar V^3}{2}}$$

for flows ⊥ to cross section, $\vec V$ & $\hat n$ are //

=> $\vec V \cdot \vec n = V$ (module)

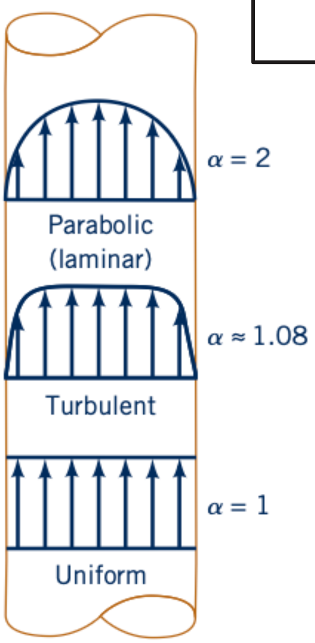
+ ρ = cost :

$$\Downarrow$$

$$\alpha = \frac{\cancel{\rho}\int_A \frac{V^3}{\cancel{2}} dA}{\cancel{\rho}\, \frac{\bar V^3}{2} \cancel{2}}$$

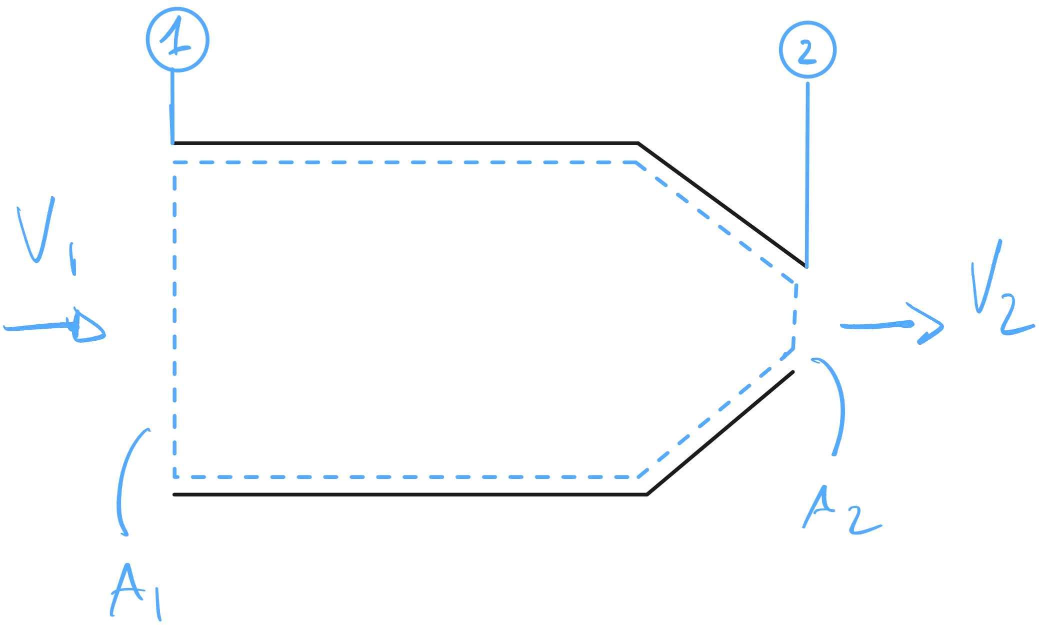
$$\boxed{\alpha = \frac{\int_A V^3 dA}{A \vec V^3}}$$

often seen this way too.



- if V = costant = $\bar V$ => $\alpha = 1$
- generally $\alpha > 1$
  - LAMINAR FLOW $\alpha = 2$
  - TURBULENT FLOW $\alpha = 1.03 - 1.10$

EXAMPLE 3.4.1 Determine an expression based upon the energy concept that relates the pressures at the upstream and downstream ends of a nozzle assuming steady flow, neglecting change in internal energy, and assuming $dH/dt = 0$ and $dW_s/dt = 0$.



SOLUTION Using the energy equation (3.4.20) for steady flow yields

$$\frac{dH}{dt} - \frac{dW_s}{dt} = \sum_{CS}\left(\frac{p}{\rho} + e_u + \frac{1}{2}V^2 + gz\right)\rho\,\mathbf{V}\cdot\mathbf{A}$$

Neglecting dH/dt and dW_s/dt the above energy equation can be expressed as

$$\int_{A_2}\left(\frac{p_2}{\rho} + e_{u_2} + \frac{1}{2}V_2^2 + gz_2\right)\rho\, V_2\, dA_2 - \int_{A_1}\left(\frac{p_1}{\rho} + e_{u_1} + \frac{1}{2}V_1^2 + gz_1\right)\rho\, V_1 dA_1 = 0$$

which can be modified to

$$\int_{A_2}\left(\frac{p_2}{\rho} + e_{u_2} + gz_2\right)\rho\, V_2 dA_2 + \int_{A_2}\frac{\rho\, V_2^3}{2}\,dA_2 - \int_{A_1}\left(\frac{p_1}{\rho} + e_{u_1} + gz_1\right)\rho\, V_1\, dA_1 - \int_{A_1}\frac{\rho\, V_1^3}{2}\,dA_1 = 0$$

For hydrostatic conditions, $\left(\frac{p}{\rho} + e_u + gz\right)$ is constant across the system, which allows these terms to be taken outside the integral:

$$\left(\frac{p_2}{\rho} + e_{u_2} + gz_2\right)\int_{A_2}\rho\, V_2\, dA_2 + \int_{A_2}\frac{\rho\, V_2^3}{2}\,dA_2 - \left(\frac{p_1}{\rho} + e_{u_1} + gz_1\right)\int_{A_1}\rho V_1\, dA_1 - \int_{A_1}\frac{\rho\, V_1^3}{2}\,dA_1 = 0$$

The term $\int \rho V dA$ is the mass rate of flow, $\dot{m}$, and the term $\int \frac{\rho\, V^3}{2}dA = \dot{m}\frac{V^2}{2}$, so

$$\left(\frac{p_2}{\rho} + e_{u_2} + gz_2\right)\dot{m} + \dot{m}\frac{V_2^2}{2} - \left(\frac{p_1}{\rho} + e_{u_1} + gz_1\right)\dot{m} - \dot{m}\frac{V_1^2}{2} = 0$$

Dividing through by $\dot{m}g$ and rearranging yields

$$\frac{p_1}{\rho\, g} + \frac{e_{u_1}}{g} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\rho\, g} + \frac{e_{u_2}}{g} + z_2 + \frac{V_2^2}{2g}$$

$\gamma = \rho g$ and rearranging yields

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + \frac{e_{u_2} - e_{u_1}}{g}$$

Neglecting changes in internal energy, $(e_{u_2} - e_{u_1})/g = 0$

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

Assuming the control volume is horizontal, $z_1 = z_2$, then

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

This energy equation relates the pressures assuming steady flow, $z_1 = z_2$, neglecting change of internal energy in the fluid and assuming $dH/dt = 0$ and $dW_s/dt = 0$.

EXAMPLE 3.4.2

For a nozzle, determine the pressure change through the nozzle between the upstream and downstream end of the nozzle. Assume steady flow, neglect changes in internal energy of the fluid, assume $dH/dt = 0$ and $dW_s/dt = 0$, and say that the nozzle is horizontal. Assume the temperature is 20°C. The velocities at the entrance and exit are $V_2 = 2.55$ m/s and $V_1 = 1.13$ m/s, respectively.

SOLUTION

Using the energy equation derived in example 3.4.1 yields

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

$$\begin{aligned} p_1 - p_2 &= \left(V_2^2 - V_1^2\right)\frac{\gamma}{2g} \\ &= \left[(2.55)^2 - (1.13)^2\right] \times \frac{9.79 \text{ kN/m}^3}{2 \times 9.81 \text{ m/s}^2} \\ &= (5.226 \text{ m}^2/\text{s}^2)(0.499 \text{ kN s}^2/\text{m}^4) \\ &= 2.608 \text{ kN/m}^2 = 2.608 \text{ kPa} = 2608 \text{ Pa} \end{aligned}$$

The pressure change is a pressure decrease of 2608 Pa.

We will see several applications of the energy conservation in the next module, Pipe Flow.

3. Newton's Second Law — the Linear Momentum Equation

This equation is based on Newton's 2nd Law of Motion:

$$\boxed{F = m a}$$

or in other words

$$m \frac{d\vec{V}}{dt} = \vec{F}$$ it's a vectorial eq.

$$m = cost \Rightarrow \frac{d m\vec{V}}{dt} = \vec{F}$$

where $m\vec{V}$ is called MOMENTUM

$\Rightarrow$ For a small particle of fluid of mass $dm = \rho\, dV$, the momentum can be expressed as $\vec{V} \rho\, dV$

So for a SYSTEM:

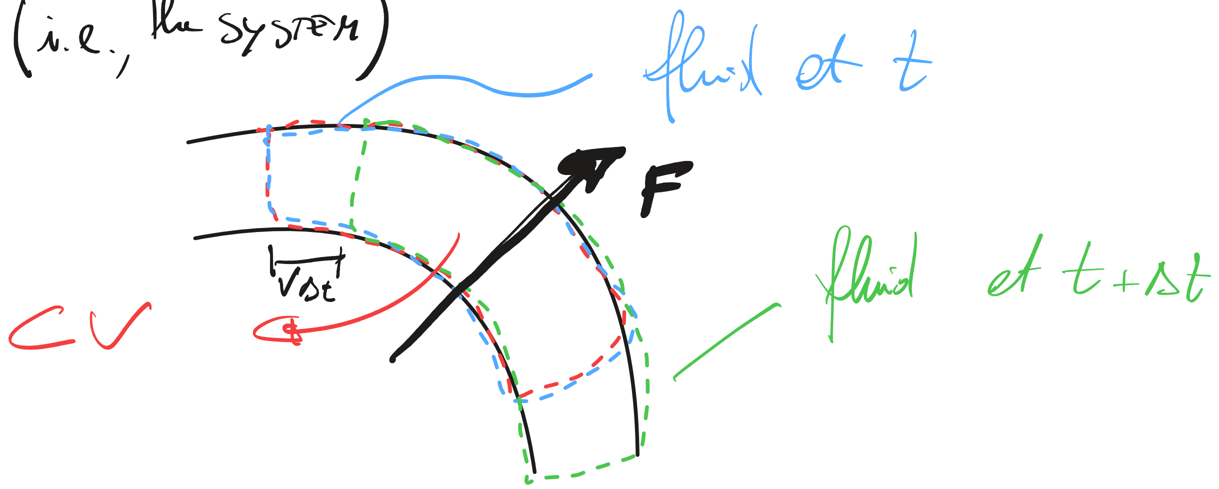
$$\boxed{\frac{D}{Dt}\int_{sys} \vec{V}\rho \, d\mathcal{V} = \sum \vec{F}_{sys}}$$

TIME RATE OF CHANGE of the LINEAR MOMENTUM OF THE SYSTEM = SUM OF ALL THE EXTERNAL FORCES ACTING ON THE SYSTEM

To go from the system view to the view of a control volume ⟹ REYNOLDS TRANSPORT THEOREM (RTT)

why do we do this? Just bc it makes things easier

E.g. it's easier to calculate the forces on a curved pipe in a fixed volume than for a moving chunk of fluid (i.e., the SYSTEM)



$\Rightarrow$ Apply RTT with:

- $B_{sys} = mV \longrightarrow$ MOMENTUM of the SYSTEM
- $b = \frac{B}{m} = V$

$$\underbrace{\frac{D}{Dt}\int_{sys} \vec{V}\rho \, dV}_{\text{TIME RATE OF CHANGE OF THE LINEAR MOMENTUM OF THE SYSTEM}} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \vec{V}\rho \, dV}_{\text{TIME RATE OF CHANGE OF LINEAR MOMENTUM OF THE CONTENTS OF CV}} + \underbrace{\int_{CS} \vec{V}\rho \vec{V}\cdot\hat{n} \, dA}_{\text{NET RATE OF FLOW OF LINEAR MOMENTUM THROUGH CS}}$$

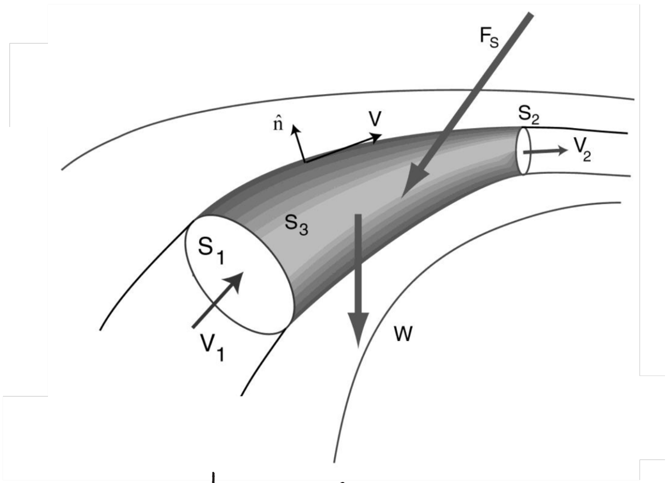
Since we're considering system $\doteq$ CV, the forces acting on the system = forces on CV

$$\sum \vec{F}_{sys} = \sum \vec{F}_{\substack{\text{content}\\ \text{of CV}}}$$

$\Rightarrow$ Combining this with Newton 2^{nd} law:

$$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho \vec{V} \, dV + \int_{CS} \vec{V}\rho\vec{V}\cdot\hat{n} \, dA = \sum \vec{F}_{\substack{\text{content of}\\ \text{CV}}}}$$

Note:



1) $\sum F_{CV}$ is the sum of all the forces ACTING on CV:

→ SURFACE FORCES
forces resulting from the surrounding ACTING on the CS

→ BODY FORCES ⇒ the weight.

$$\Rightarrow \boxed{\sum \vec{F}_{CV} = \sum \vec{F}_s + \vec{W}}$$

2) The equation is VECTORIAL ⇒ the direction matters and there is an equation per COORDINATE!

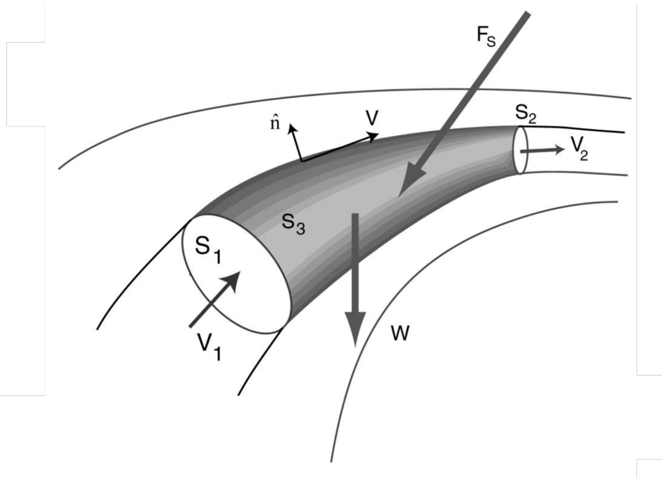
3) The integral over the CONTROL SURFACE is the NET MOMENTUM FLUX ACROSS the control surface of the fluid entering and/or leaving the CV!

⇒ ASSUMING V to be the average velocity across the SURFACE:

FLUX in or out

$$\int_{CS} \vec{V} \rho \vec{V} \cdot \hat{n} \, dA = \vec{V} \rho \int_{CS} \vec{V} \cdot \hat{n} \, dA = \rho \vec{V} Q$$

velocity VECTOR with the sign depending on the DIRECTION with respect to the AXIS orientation!



For example here:

$$\rho Q(\vec{V}_2 - \vec{V}_1)\,!$$

4) For steady flow $\Rightarrow \frac{\partial}{\partial t} = 0$!

so the equation reduces to:

$$\boxed{\sum \vec{F}_S + \vec{W} = \rho Q(\vec{V}_2 - \vec{V}_1)}$$

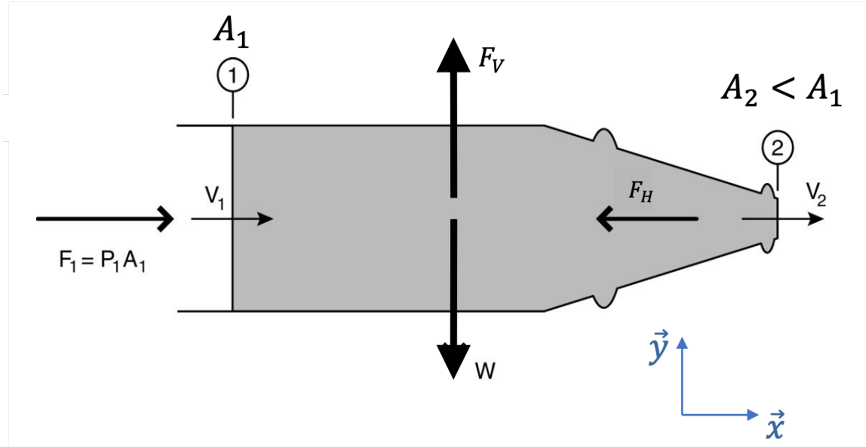
5) For steady AND STATIC FLOW $\left(\frac{\partial}{\partial t} = 0 \text{ AND } \vec{V} = 0\right)$ the momentum equation reduces to a balance of FORCES!

$$\boxed{\sum \vec{F}_s + \vec{W} = \sum \vec{F}_{TOT} = 0}$$

$\Rightarrow$ we will see this in HYDROSTATICS!

Example 1

What is the value of the force FH required to keep the hose fixed in its position? We work in 2D (axis x-y).



We assume STEADY CONDITIONS $\Rightarrow \frac{\partial}{\partial t} = 0$

$\Downarrow$

$$\sum \vec{F_S} + \vec{W} = \int_{CS} \rho \vec{V} \vec{V} \cdot \hat{n} \, dA$$

Let's split this in the two components:

$$x: \quad \sum \vec{F_S} = \int_{CS} \rho \vec{V} \vec{V} \cdot \hat{n} \, dA$$

there are only Surface forces in x and the momentum flux through the two openings of the hose

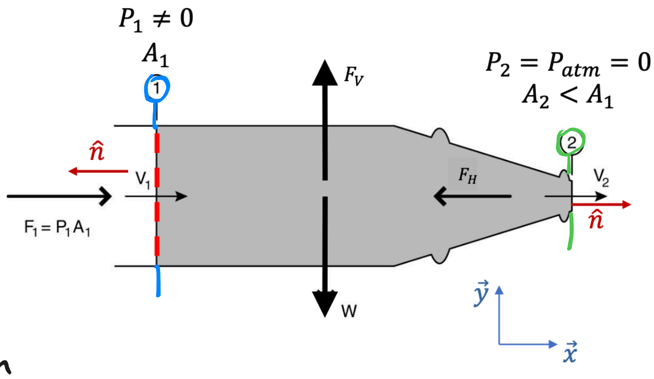
$$y: \quad \vec{F_V} - \vec{W} = \int_{CS} \rho \vec{V} \vec{V} \cdot \hat{n} \, dA$$

no V in y!

$$\vec{F_V} = \vec{W}$$

$x: \sum \vec{F}_s = \int_{CS} \rho \vec{V} \vec{V} \cdot \hat{n} \, dA$

we now separate this in the two control surface ① & ② where the flux is flowing through



$$\sum \vec{F}_s = \int_{S1} \rho \vec{V}_1 \underbrace{\vec{V}_1 \cdot \hat{n}}_{|V_1||n|\cos\theta \,=\, -V_1} \, ds_1 + \int_{S2} \rho \vec{V}_2 \underbrace{\vec{V}_2 \cdot \hat{n}}_{|V_2||n|\cos\theta \,=\, +V_2} \, ds_2$$

→ θ is the angle between $\vec{V}$ and $\vec{n}$!

$$F_1 - F_H = \rho (+\vec{V}_1)(-Q) + \rho(+\vec{V}_2)(+Q)$$

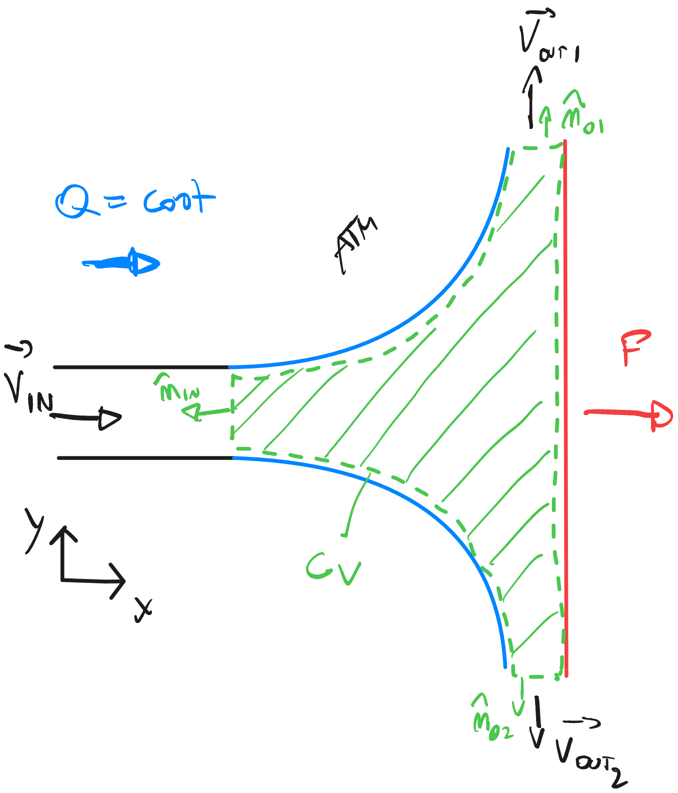
the velocity is in the same direction as X so POSITIVE

the sign of the flux instead is determined by $\vec{V} \cdot \hat{n}$!

$$\boxed{F_H = F_1 - \rho Q (V_2 - V_1)}$$

Example 2

Let's calculate the force of a water jet impinging on a flat vertical wall. Neglect the weight and atmosferic pressure. Assume steady state (Q=constant).



$$\vec V_{in} = (u_1, 0, 0)$$
$$\vec V_{OUT1} = (0, V_{01}, 0)$$
$$\vec V_{OUT2} = (0, V_{02}, 0)$$

by symmetry $V_{OUT1} = V_{OUT2}$

F_{WALL} = resistance of wall against the water (external surf. force)

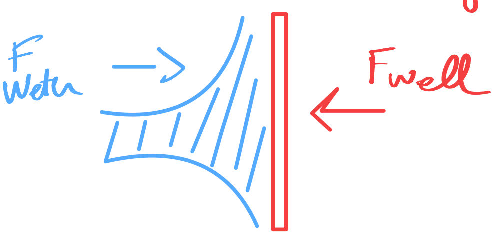
$F_{FLOW} = -F_{WALL}$

$$X: \quad \sum F_{S_x} = \int_{CS} \rho \vec V \vec V \cdot \hat n$$

$$F_{wall} = \rho V_{in} \int_{CS} |V_{in}| |n| \cos\theta$$

$$\stackrel{!}{=} -\rho V_{in} Q$$

the negative sign indicates that the direction of F that I originally assumed is WRONG. Which makes sense since the wall is resisting the force of water!



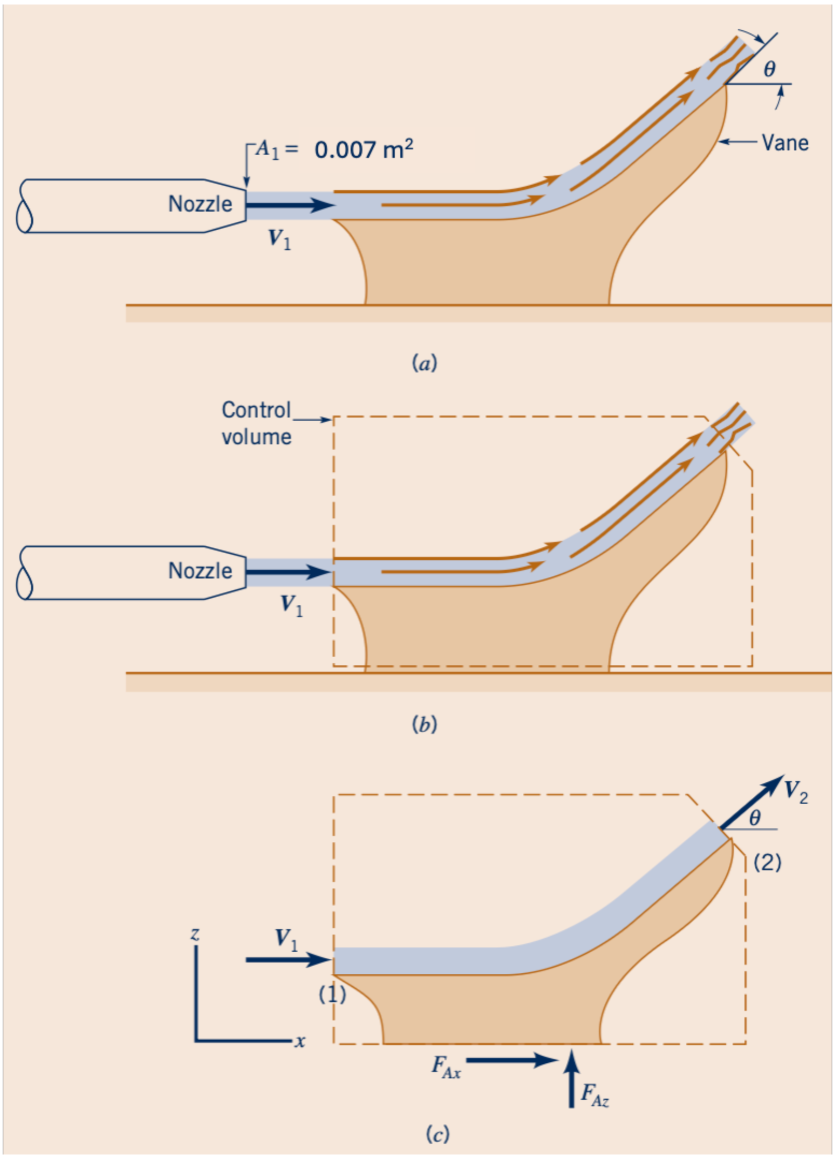
$$|F_{wall}| = |F_{water}|$$

$$\vec F_{wall} = -\vec F_{water}$$

$$\Rightarrow F_{FLW} = \rho \vec{V}_1 \vec{Q} = \rho u^2 A$$

$$y: \quad \sum F_{S_y} = 0 = \rho V_{01}^2 A - \rho V_{02} A = 0$$

Example 3



As shown in Figure here (panel a), a horizontal jet of water exits a nozzle with a uniform speed of V1 3 m/s, strikes a vane, and is turned through an angle θ. The density ρ=1000 kg/m^3

Determine the anchoring force needed to hold the vane stationary if gravity and viscous effects are negligible.

We select a control volume that includes the vane and a portion of the water (see Figure panels b, c) and apply the linear momentum equation to this fixed control volume. The only portions of the control surface across which fluid flows are section (1) (the entrance) and section (2) (the exit). Hence, the x and z components become:

For x: $$\cancel{\frac{\partial}{\partial t}\int_{CV} \rho u \, dV} + \int_{CS} u \rho \vec{V} \cdot \hat{n} \, dA = \sum \vec{F}_x$$

0 because the flow is steady!

For z: $$\cancel{\frac{\partial}{\partial t}\int_{CV} \rho w \, dV} + \int_{CS} w \rho \vec{V} \cdot \hat{n} \, dA = \sum \vec{F}_z$$

0 because flow is steady

where: $\vec{V} = u\hat{i} + w\hat{k}$ and $\sum F_x$ and $\sum F_z$ are the net x and z components of the force acting on CV

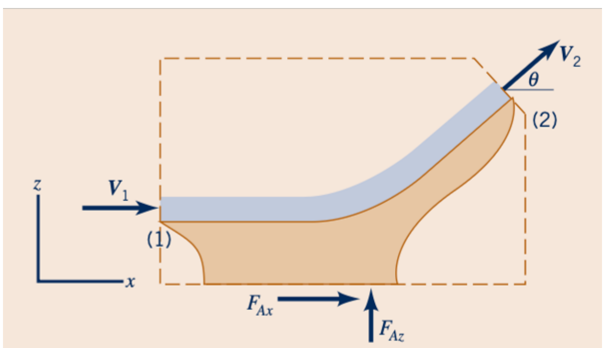
$\Rightarrow$ which means:

$$x: \quad u_2 \rho A_2 V_2 - u_1 \rho A_1 V_1 = \sum F_x$$

$$z: \quad w_2 \rho A_2 V_2 - w_1 \rho A_1 V_1 = \sum F_z$$

Depending on the particular flow situation being considered and the coordinate system chosen, the x and z components of velocity, u and w, can be positive, negative, or zero. In this example the flow is in the positive direction at both the inlet and the outlet.

The water enters and leaves the control volume as a free jet at atmospheric pressure. Hence, there is atmospheric pressure surrounding the entire control volume, and the net pressure force on the control volume surface is zero. If we neglect the weight of the water and vane, the only forces applied to the control volume contents are the horizontal and vertical components of the anchoring force, F_Ax and F_Az, respectively which we write like in the image but we do not know the orientation yet.



With negligible gravity and viscous effects, and since p_1=p_2, the speed of the fluid remains constant so that V_1=V_2=3 m/s (see the Bernoulli equation). Hence, at section (1), u_1=V_1, w_1=0, and at section (2), u_2=V_1 cosθ, w_2=V_1 sinθ.

Using conservation of mass $\Rightarrow A_1 V_1 = A_2 V_2 \Rightarrow \underline{A_1 = A_2}$

$$\Downarrow$$

$$\rightarrow F_{Ax} = -V_1 \rho A_1 V + V_1 \cos\theta \rho A_2 V_1$$

$$= \rho A_1 V_1^2 (1 - \cos\theta)$$

$$\rightarrow F_{Az} = \rho A_1 V_1^2 \sin\theta$$

with the given data:

$$F_{Ax} = -1000 \frac{kg}{m^3} (0.007\, m^2) \left(3 \frac{m}{s}\right)^2 (1 - \cos\theta) = -63 (1 - \cos\theta) \; [N]$$

$$F_{Az} = 1000 \frac{kg}{m^3} (0.007\, m^2) \left(3 \frac{m}{s}\right)^2 (\sin\theta) = 63 \sin\theta \; [N]$$

Note that if θ=0 (i.e., the vane does not turn the water), the anchoring force is zero. The inviscid fluid merely slides along the vane without putting any force on it. If θ=90°, then F_Ax=-63 N and F_Az=63 N. It is necessary to push on the vane (and, hence, for the vane to push on the water) to the left (F_Ax is negative) and up in order to change the direction of flow of the water from horizontal to vertical. This momentum change requires a force. If θ=180°, the water jet is turned back on itself. This requires no vertical force (F_Az=0), but the horizontal force (F_Ax=-126 N) is two times that required if θ=90° . This horizontal fluid momentum change requires a horizontal force only.